Transformasi Laplace Fungsi Step

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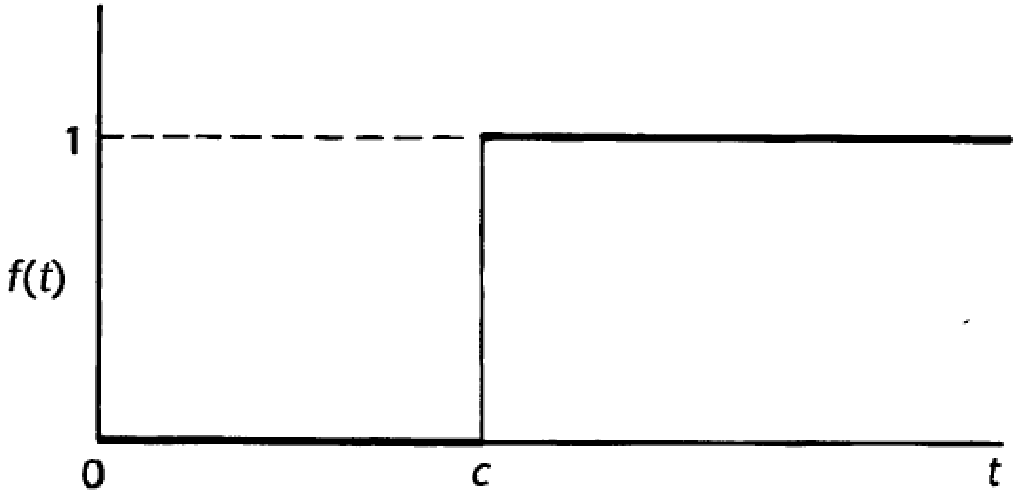
Pengertian Fungsi Step

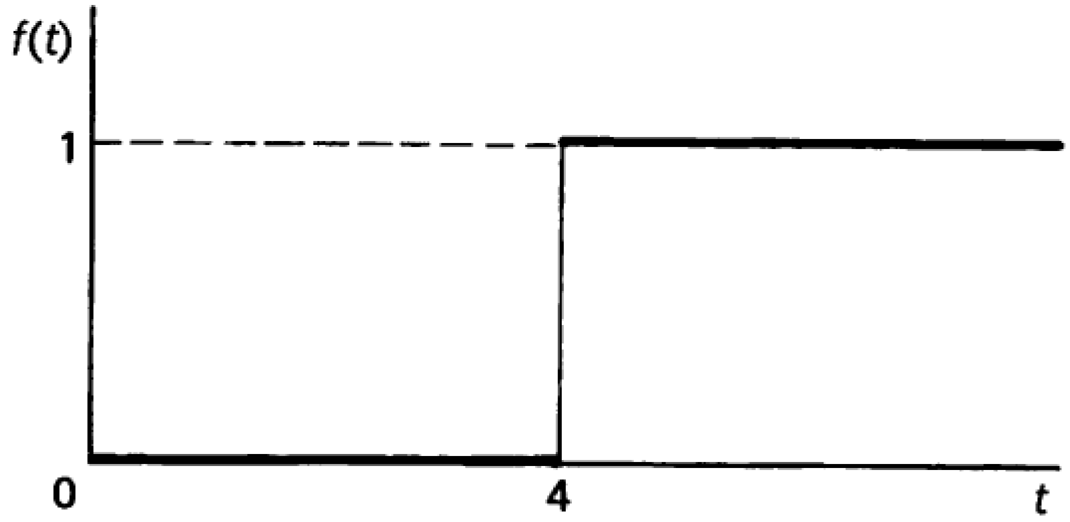
- Fungsi step adalah fungsi yang bernilai nol pada saat $t < c$ dan bernilai 1 pada saat $t \geq c$



$$f(t) = 0 \quad \text{for } t < c$$
$$f(t) = 1 \quad \text{for } t \geq c$$

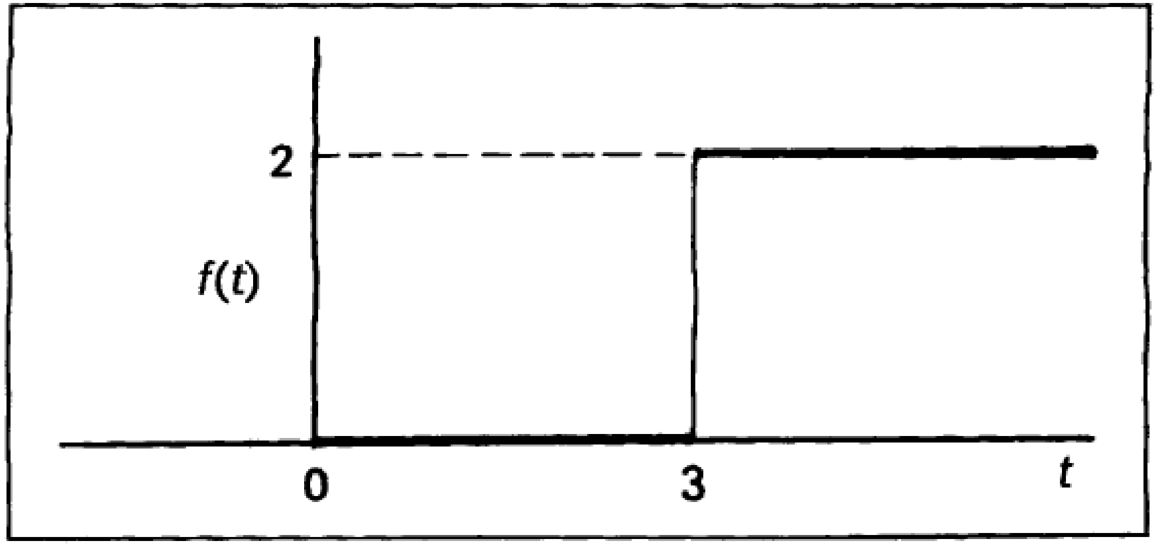
- Fungsi ini dinotasikan dengan $f(t) = u(t - c)$
- Dimana c merupakan nilai t yang menandai perubahan nilai fungsi dari 0 ke 1

- Dengan demikian, maka fungsi berikut dinyatakan sebagai..



$$f(t) = u(t-4)$$

- Serupa dengan diatas, grafik dari:
- $f(t) = 2u(t-3)$ adalah...

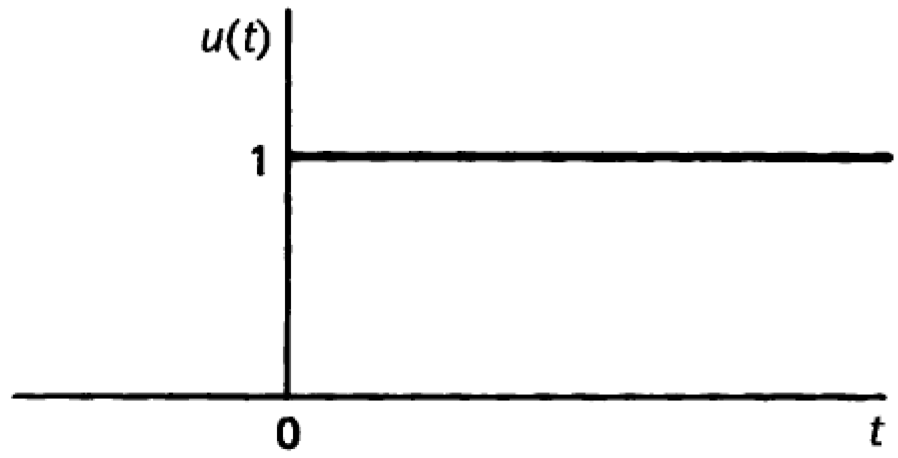


Berarti fungsi step $u(t-c)$ hanya mempunyai dua nilai yaitu: untuk $t < c$, $u(t-c) = 0$

dan $t \geq c$, $u(t-c) = 1$

Fungsi step pada titik awal

- Jika fungsi step terjadi pada titik awal (titik 0), berarti c = 0, maka fungsi f(t) = u(t – c) menjadi f(t) = u(t)

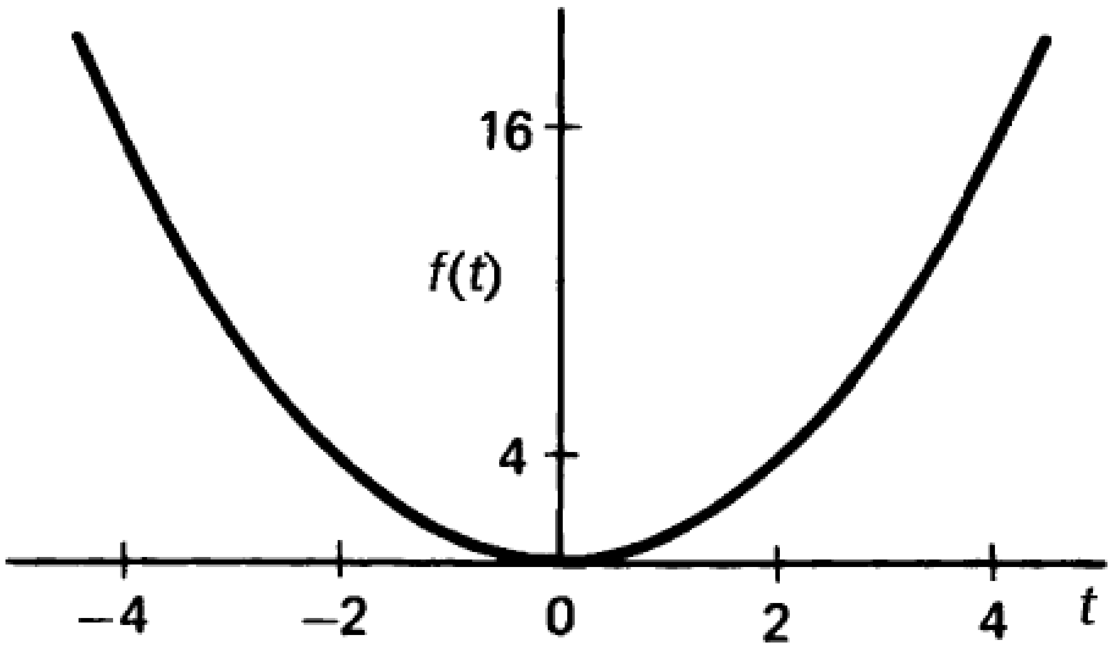


i.e. $u(t) = 0$ for $t < 0$

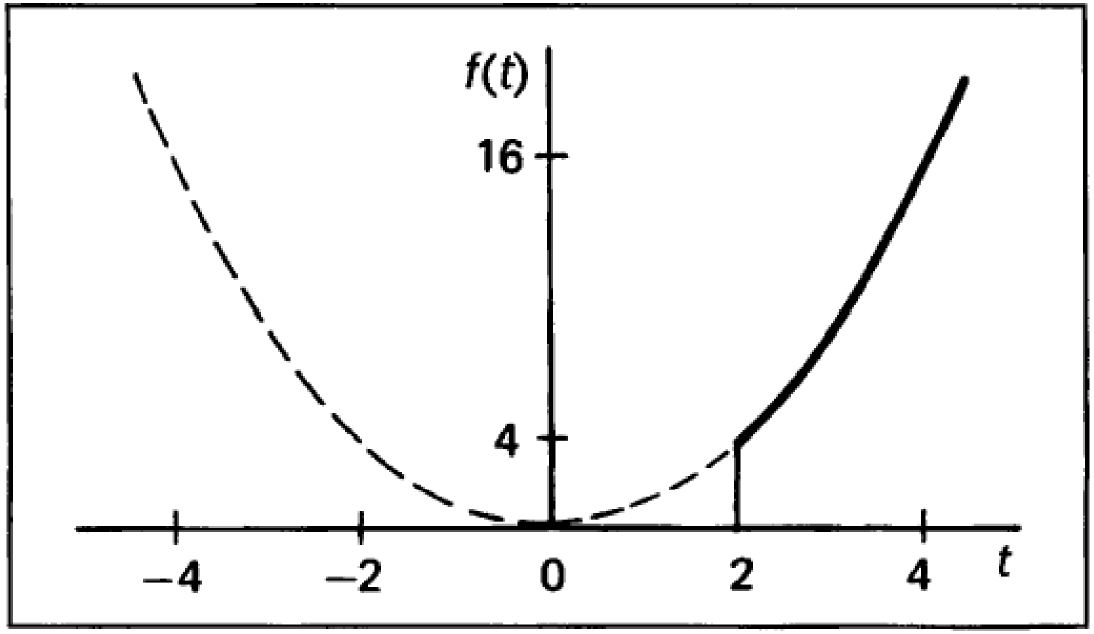
$u(t) = 1$ for $t \geq 0$.

Pengaruh dari Fungsi Step

- Berikut ini adalah fungsi $f(t) = t^2$



- Grafik dari f(t) = u(t-2).t^2 adalah...



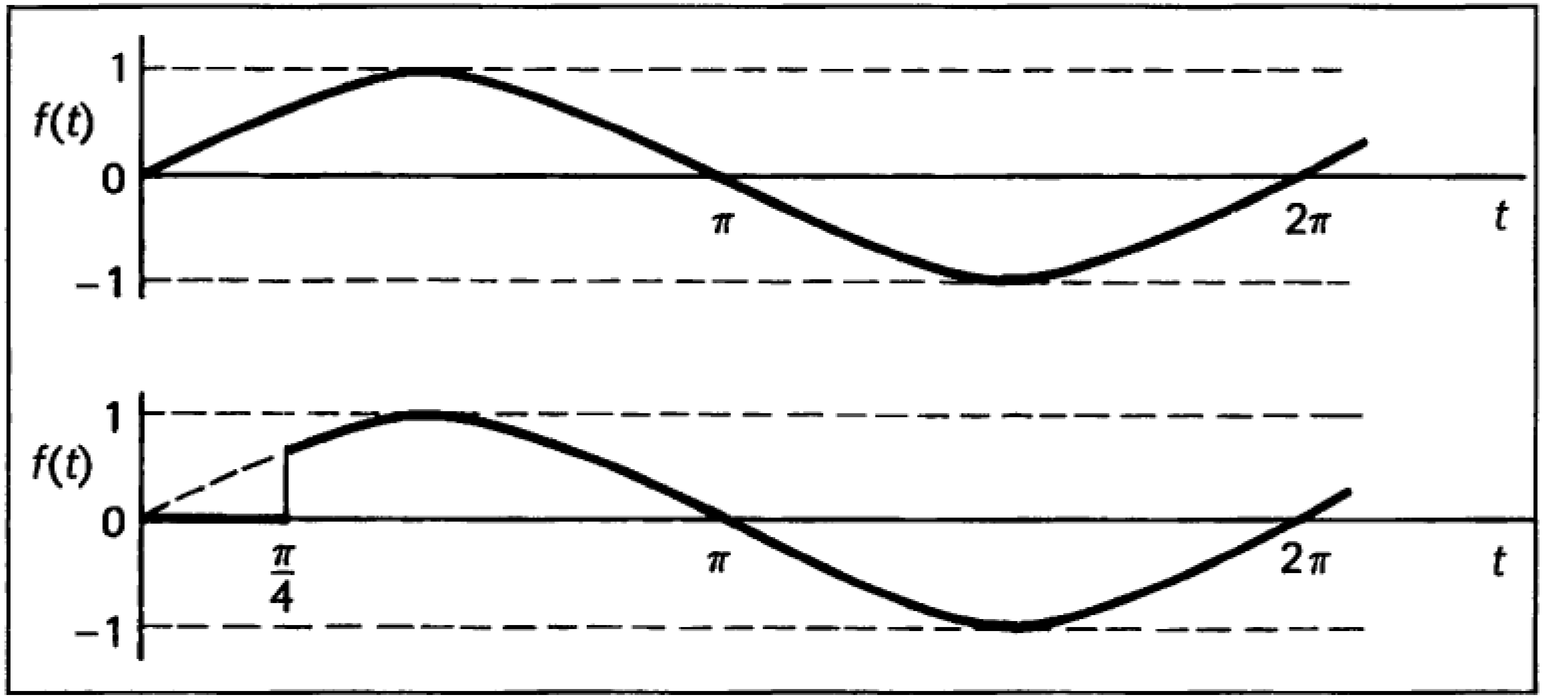
$$\text{For} \quad t < 2,\ u(t-2) = 0 \quad \therefore\ u(t-2)\cdot t^2 = 0\cdot t^2 = 0$$
$$t \geq 2,\ u(t-2) = 1 \quad \therefore\ u(t-2)\cdot t^2 = 1\cdot t^2 = t^2$$

- Berarti fungsi u(t-2) menghilangkan fungsi t^2 utk semua nilai t sampai t =2 dan mengembalikan fungsi t^2 pada t =2

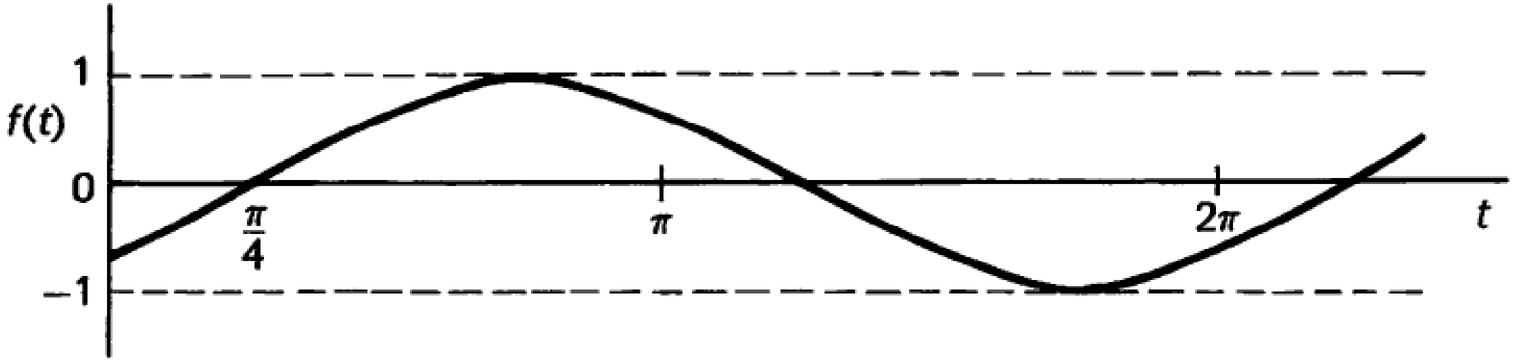
- Sekarang coba gambarkan grafik fungsi:

(a) $f(t) = \sin t \quad \text{for } 0 < t < 2\pi$

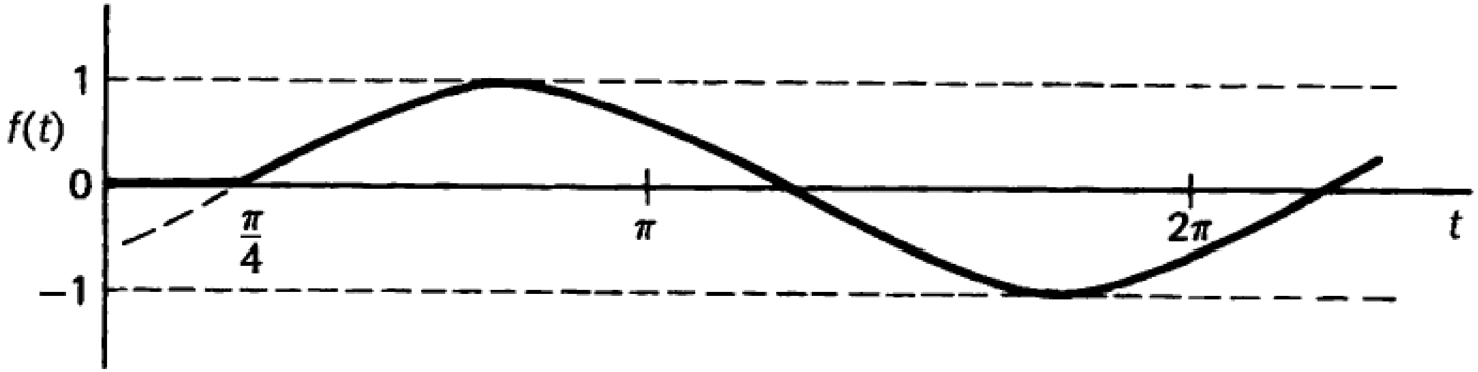
(b) $f(t) = u(t - \pi/4) \cdot \sin t \quad \text{for } 0 < t < 2\pi.$



- Gambarkan grafik f(t) = sin (t - π/4) adalah sbb:



- Sedangkan grafik $f(t) = u(t - \pi/4) \cdot \sin(t - \pi/4)$ adlh:



- Grafik tsb jg merupakan grafik f(t) = sin t utk t > 0 yg digeser sejauh π/4

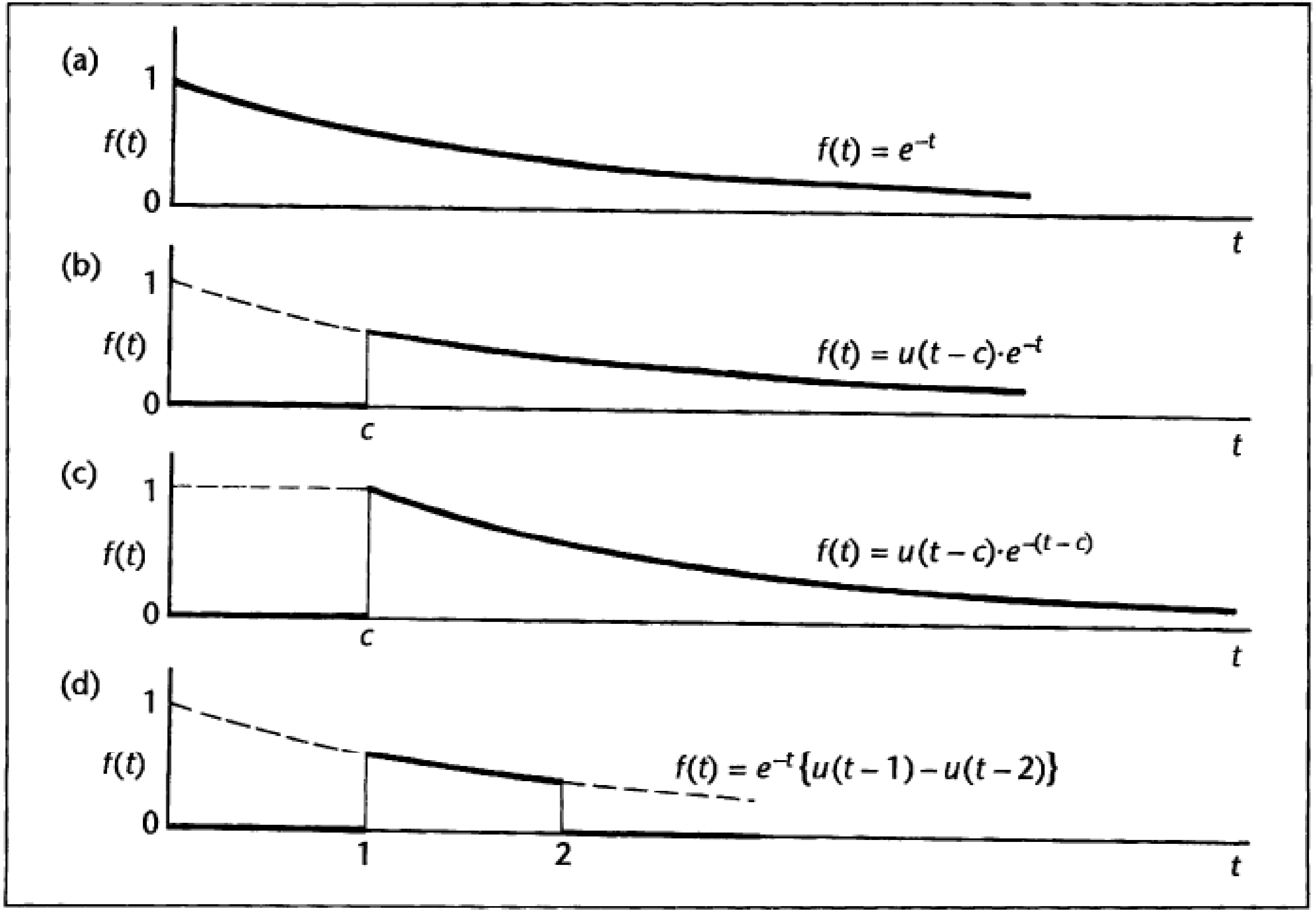
- Secara umum, grafik f(t) = u(t-c). Sin (t-c) adalah grafik f(t)= sin t (t > 0) yang digeser sepanjang sumbu t sejauh c.
- Dgn aturan tsb, untuk t > 0 gambarkan grafik dari:

(a) $f(t) = e^{-t}$

(b) $f(t) = u(t-c) \cdot e^{-t}$

(c) $f(t) = u(t-c) \cdot e^{-(t-c)}$

(d) $f(t) = e^{-t}\{u(t-1) - u(t-2)\}$.



(a)
1
f(t)
0
f(t) = e^{-t}
t
(b)
1
f(t)
0
c
f(t) = u(t − c)·e^{-t}
t
(c)
1
f(t)
0
c
f(t) = u(t − c)·e^{-(t − c)}
t
(d)
1
f(t)
0
1
2
f(t) = e^{-t}{u(t − 1) − u(t − 2)}
t

Transformasi Laplace dari u(t-c)

$$L\{u(t-c)\} = \int_0^\infty e^{-st} u(t-c)\, \mathrm{d}t$$

but

$$e^{-st}u(t-c) = \begin{cases} 0 & \text{for } 0 < t < c \\ e^{-st} & \text{for } t \geq c \end{cases}$$

so that

$$L\{u(t-c)\} = \int_0^\infty e^{-st} u(t-c)\, \mathrm{d}t = \int_c^\infty e^{-st}\, \mathrm{d}t$$

$$= \left[\frac{e^{-st}}{-s}\right]_c^\infty = \frac{e^{-sc}}{s}$$

- Sehingga transformasi laplace dari fungsi step pada titik awal adalah:

$$L\{u(t)\} = \ldots\ldots\ldots\ldots\ldots$$

Because $c = 0$.

So $$L\{u(t-c)\} = \frac{e^{-cs}}{s}$$

and $$L\{u(t)\} = \frac{1}{s}.$$

Laplace transform of $u(t-c)\cdot f(t-c)$ (the second shift theorem)

Because

$$L\{u(t-c)\cdot f(t-c)\} = \int_0^\infty e^{-st}u(t-c)\cdot f(t-c)\,\mathrm{d}t$$

$$\text{but } e^{-st}u(t-c) = \begin{cases} 0 & \text{for } 0 < t < c \\ e^{-st} & \text{for } t \geq c \end{cases}$$

so that

$$L\{u(t-c)\cdot f(t-c)\} = \int_c^\infty e^{-st} f(t-c)\,\mathrm{d}t$$

$$L\{u(t-c)\cdot (f(t-c)\} = e^{-cs}L\{f(t)\} = e^{-cs}F(s)$$

So $L\left\{u(t-4)\cdot(t-4)^2\right\} = e^{-4s}\cdot F(s)$ where $F(s) = L\{t^2\}$

$$= e^{-4s}\left(\frac{2!}{s^3}\right) = \frac{2e^{-4s}}{s^3}$$

Note that $F(s)$ is the transform of t^2 and *not* of $(t-4)^2$.

In the same way:

$L\{u(t-3)\cdot\sin(t-3)\} = e^{-3s}\cdot F(s)$ where $F(s) = L\{\sin t\}$

$$= \frac{1}{s^2+1}$$

$$\therefore\ L\{u(t-3)\cdot\sin(t-3)\} = e^{-3s}\left(\frac{1}{s^2+1}\right)$$

Latihan

So now do these in the same way.

(a) $L\left\{u(t-2)\cdot(t-2)^3\right\} = \dots\dots\dots\dots$

(b) $L\{u(t-1)\cdot\sin 3(t-1)\} = \dots\dots\dots\dots$

(c) $L\{u(t-5)\cdot e^{(t-5)}\} = \dots\dots\dots\dots$

(d) $L\{u(t-\pi/2)\cdot\cos 2(t-\pi/2)\} = \dots\dots\dots\dots$

Jawab

(a) $L\left\{u(t-2)\cdot(t-2)^3\right\} = e^{-2s}\cdot F(s) \quad \text{where } F(s) = L\{t^3\}$

$$= e^{-2s}\left(\frac{3!}{s^4}\right) = \frac{6e^{-2s}}{s^4}$$

(b) $L\{u(t-1)\cdot \sin 3(t-1)\} = e^{-s}\cdot F(s) \quad \text{where } F(s) = L\{\sin 3t\}$

$$= e^{-s}\left(\frac{3}{s^2+9}\right) = \frac{3e^{-s}}{s^2+9}$$

(c) $L\left\{u(t-5)\cdot e^{(t-5)}\right\} = e^{-5s}\cdot F(s) \quad \text{where } F(s) = L\{e^t\}$

$$= e^{-5s}\left(\frac{1}{s-1}\right) = \frac{e^{-5s}}{s-1}$$

(d) $L\{u(t-\pi/2)\cdot \cos 2(t-\pi/2)\} = e^{-\pi s/2}\cdot F(s) \quad \text{where } F(s) = L\{\cos 2t\}$

$$= e^{-\pi s/2}\left(\frac{s}{s^2+4}\right) = \frac{s\cdot e^{-\pi s/2}}{s^2+4}$$

So $L\{u(t-c)\cdot f(t-c)\} = e^{-cs}\cdot F(s) \quad \text{where } F(s) = L\{f(t)\}$.

Written in reverse, this becomes

$$\text{If } F(s) = L\{f(t)\}, \text{ then } e^{-cs} \cdot F(s) = L\{u(t-c) \cdot f(t-c)\}$$

where c is real and positive.

This is known as the *second shift theorem.*

Example 1

Find the function whose transform is $\dfrac{e^{-4s}}{s^2}$.

The numerator corresponds to e^{-cs} where $c = 4$ and therefore indicates $u(t-4)$.

Then $\dfrac{1}{s^2} = F(s) = L\{t\} \quad \therefore\ f(t) = t$.

$$\therefore\ L^{-1}\left\{\frac{e^{-4s}}{s^2}\right\} = u(t-4) \cdot (t-4)$$

Example 2

Determine $L^{-1}\left\{\frac{6e^{-2s}}{s^2+4}\right\}$.

The numerator contains e^{-2s} and therefore indicates $\boxed{u(t-2)}$.

The remainder of the transform, i.e. $\frac{6}{s^2+4}$, can be written as $3\left(\frac{2}{s^2+4}\right)$

$$\therefore \quad \frac{6}{s^2+4} = F(s) = L\{\ldots 3\sin 2t \ldots\}$$

$$\therefore \quad L^{-1}\left\{\frac{6e^{-2s}}{s^2+4}\right\} = u(t-2)\cdot f(t-2) \quad \text{where } f(t) = L^{-1}\left\{\frac{6}{s^2+4}\right\}$$

$$= u(t-2)\cdot 3\sin 2(t-2)$$

Example 3

Determine $L^{-1}\left\{\frac{s \cdot e^{-s}}{s^2+9}\right\}$.

Because the numerator contains e^{-s} which indicates $u(t-1)$.

Also $\frac{s}{s^2+9} = F(s) = L\{\cos 3t\}$

$\therefore\ f(t) = \cos 3t \quad \therefore\ f(t-1) = \cos 3(t-1)$.

$$\therefore\ L^{-1}\left\{\frac{s \cdot e^{-s}}{s^2+9}\right\} = u(t-1) \cdot \cos 3(t-1)$$

Exercise

Determine the inverse transforms of the following.

(a) $\dfrac{2e^{-5s}}{s^3}$

(b) $\dfrac{3e^{-2s}}{s^2-1}$

(c) $\dfrac{8e^{-4s}}{s^2+4}$

(d) $\dfrac{2s \cdot e^{-3s}}{s^2-16}$

(e) $\dfrac{5e^{-s}}{s}$

(f) $\dfrac{s \cdot e^{-s/2}}{s^2+2}$

Results – all very straightforward.

(a) $u(t-5)\cdot(t-5)^2$

(b) $3u(t-2)\cdot\sinh(t-2)$

(c) $4u(t-4)\cdot\sin 2(t-4)$

(d) $2u(t-3)\cdot\cosh 4(t-3)$

(e) $5u(t-1)$

(f) $u(t-1/2)\cdot\cos\sqrt{2}(t-1/2)$.